Laplace :-

A) Transform

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = f(s)$

Standard formula

1) $L\{1\} = \frac{1}{s}$

2) $L\{e^{at}\} = \frac{1}{s-a}$

3) $L\{e^{-at}\} = \frac{1}{s+a}$

4) $L\{\sin at\} = \frac{a}{s^2+a^2}$

5) $L\{\sinh at\} = \frac{a}{s^2-a^2}$

6) $L\{\cos at\} = \frac{s}{s^2+a^2}$

7) $L\{\cosh at\} = \frac{s}{s^2-a^2}$

8) $L\{t^n\} = \frac{n!}{s^{n+1}}$ or $\frac{\Gamma(n+1)}{s^{n+1}}$

9) $L\{c^{at}\} = \frac{1}{s - a\log c}$

Other Trigonometric

1) $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

2) $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

3) $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

4) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

5) $\sin^2 x = \frac{1-\cos 2x}{2}$

6) $\cos^2 x = \frac{1+\cos 2x}{2}$

6) $\sin^3 x = \frac{1}{4}[3\sin x - \sin 3x]$

$\cos^3 x = \frac{1}{4}[3\cos x + \cos 3x]$

7) $\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

8) $\sinh x = \frac{e^x - e^{-x}}{2}$ $\cosh x = \frac{e^x + e^{-x}}{2}$

Types:-

1) first shifting prop.

$L\{e^{at} f(t)\} = f(s-a)$

$L\{e^{-at} f(t)\} = f(s+a)$

2) particular value

$\int_0^\infty e^{-st} f(t)\, dt$

3) Multiplication by t

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

4) Division by 't'

$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s)\, ds$

5) Laplace T of Integral

$L\left\{\int_0^t f(u)\, du\right\} = \frac{1}{s} \times f(s)$

6) Laplace T of Derivative

$L\{f(t)\} = f(s)$

$L\{f'(t)\} = sf(s) - s^0 f(0)$

$L\{f''(t)\} = s^2 f(s) - s' f(0) - s^0 f'(0)$

7) Periodic function

$f(t+P) = f(t)$ ← Hint

$\underset{\text{period}}{P}$

$$L\{f(t)\} = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$$

8) Change of scale prop.

If $L\{f(t)\} = f(s)$

then $L\{f(at)\} = \frac{1}{a} f(s/a)$

9) Second shifting property

$L\{f(t-a)\} = e^{-as} f(s)$

Laplace Inverse

1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$

2) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{+at}$

3) $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

4) $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

5) $L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$

6) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cos at$

7) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

8) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$ or $\frac{t^{n-1}}{\Gamma n}$

Types!-

1) Partial fraction

2) Convolution theorem

3) $L^{-1}\{f_1(s) \cdot f_2(s)\} = \int_0^t f_1(u) \cdot f_2(t-u)\,du$

3) Log & Inverse

$L^{-1}\{f(s)\} = -\frac{1}{t} \cdot L^{-1}\{f'(s)\}$

4) Division by 's'

$L^{-1}\left\{\frac{1}{s} \times f(s)\right\} = \int_0^t f(u)\,du.$

Types

5) Heaviside

a) L.T

$L\{H(t-a)\} = \frac{e^{-as}}{s}$

$L\{f(t-a)H(t-a)\} = e^{-as} f(s)$

b) Inverse

$L^{-1}\{e^{-as} f(s)\} = f(t-a)H(t-a)$

6) Dirac delta

i) $L\{\delta(t-a)\} = e^{-as}$

ii) $L\{\delta(t-a)f(t)\} = e^{-as} f(a)$

$= e^{-as} f(a)$

7) Application of Laplace T

Assume

$L\{y\} = Y(s)$

$L\{y'\} = sY(s) - s^0 y(0)$

$L\{y''\} = s^2 Y(s) - s^1 y(0) - s^0 y'(0)$

Fourier Series :-

interval

π : $(0, 2\pi)$ $(-\pi, \pi)$

l : $(0, 2l)$ $(-l, l)$

A) Fourier Series in 'l' interval

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int f(x)\,dx$$

$$a_n = \frac{1}{l}\int f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int f(x)\sin\frac{n\pi x}{l}\,dx$$

Fourier Series in π interval

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int f(x)\sin nx\,dx$$

B) General store

i) $\sin 0 = \sin n\pi = 0$

ii) $\cos 0 = \cos 2n\pi = (-1)^{2n} = e^{\pm 2in\pi} = 1$

iii) $\cos n\pi = e^{\pm in\pi} = (-1)^n$

iv) $\cos(1 \pm n)2\pi = 1$

$\cos(1 \pm 2n)\pi = -1$

$\cos(1 \pm n)\pi = -(-1)^n$

v) $\frac{d}{dx}$ constant $= 0$, $\int$ constant $dx = c \times x$

vi) $\frac{d}{dx} x^n = nx^{n-1}$ $\int x^n dx = \frac{x^{n+1}}{n+1}$

vii) $\int u \cdot v\,dx$: 1) u as it is , v ka integration

2) 1st ka drt , 2nd ka integration

$+ - + - - \ldots - -$

C) Types of function $[-, +]$

a) Even

$f(x) = x^2 \ (-\pi, \pi)$

put $x = -x$, $f(-x) = (-x)^2$

$= x^2$

$=$ same

f^n is even , $b_n = 0$

b) odd

$f(x) = \sin x \ (-1, 1)$

put $x = -x$, $f(-x) = \sin(-x)$

$= -\sin x$

$=$ sign change

$\therefore f^n$ is odd $a_0 = 0$

$a_n = 0$

c) NENO

$f(x) = x + x^2 \ (-\pi, \pi)$

put $x = -x$

$f(-x) = -x + (-x)^2$

$= -x + x^2$

f^n is NENO

D) $\pi^2/18$

1) If $\frac{\pi^2}{6}$ and $\frac{\pi^2}{12}$ are known, then add $\frac{\pi^2}{6}$ & $\frac{\pi^2}{12}$

2) If $\frac{\pi^2}{6}$ and $\frac{\pi^2}{12}$ are unknown, then

for continuous f^n — put $x=0$ in f.s.

for discontinuous f^n — find new functions

$$f(x) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right]$$

then put $x=0$ in f.s.

E) Parseval's Identity

$$\int f(x)^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

complex form of f.s.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \qquad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2l}\int f(x) e^{-\frac{in\pi x}{l}} dx \qquad c_n = \frac{1}{2\pi}\int f(x) e^{-inx} dx$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$e^{\pm 2in\pi} = 1 \qquad e^{\pm in\pi} = (-1)^n$$

$$\int e^{Ax}\sin Bx\, dx = \frac{e^{Ax}}{A^2+B^2}(A\sin Bx - B\cos Bx)$$

$$\int e^{Ax}\cos Bx\, dx = \frac{e^{Ax}}{A^2+B^2}(A\cos Bx + B\sin Bx)$$

Complex Variable

Basic Info:

1) Complex Number

$z = x + iy$

2) complex conjugate (i ka sign change)

$\bar{z} = x - iy$

3) Modulus

$|z| = \sqrt{x^2+y^2} = r$

4) Amplitude

$\theta = \tan^{-1} y/x$

5) function of z

$f(z) = u + iv = w$

Special formula

6) principal value

$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$

7) Euler's formula

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

8)

i) $\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$

ii) $\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$

iii) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

iv) $\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$

Imaginary Angle

$\sin ix = i\sinh x$	$\sinh ix = i\sin x$
$\cos ix = \cosh x$	$\cosh ix = \cos x$

Derivative

1) $\frac{d}{dx}\log f(x) = \frac{1}{f(x)} \cdot \frac{d}{dx} f(x)$

2) $\frac{d}{dx}\tan^{-1} f(x) = \frac{1}{1+f(x)^2} \cdot \frac{d}{dx} f(x)$

3) $\frac{d}{dx}\frac{1}{f(x)} = \frac{-1}{f(x)^2} \cdot \frac{d}{dx} f(x)$

4) $\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

5) $\frac{d}{dx} uv = u\frac{dv}{dx} + v\frac{du}{dx}$

$ds = c$	$\int s = -c$
$dc = -s$	$\int c = s$

6) $\frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}$

7) $\frac{d}{dx} f^n(x) = n\, f(x)^{n-1} \cdot \frac{d}{dx} f(x)$

Integration

1) $\int \frac{1}{x}\, dx = \log x$

2) $\int \sec^2 x\, dx = \tan x$

3) $\int \mathrm{cosec}^2 x\, dx = -\cot x$

Note!- Derivative & integration of Hyperbolic function are always positive

* condition for Analytic function

Cartesian form

$u \quad v$

$x \quad y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Polar form

$u \quad v$

$\partial r \quad r\partial\theta$

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$

* Harmonic function [Laplace eq^n]

cartension $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ & polar form

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

* i) $z \cdot \bar{z} = |z|^2 = |\bar{z}|^2$ ii) $f(z) \cdot f(\bar{z}) = |f(z)|^2 = |f(\bar{z})|^2$

iii) $f'(z) \cdot f'(\bar{z}) = |f'(z)|^2 = |f'(\bar{z})|^2$ iv) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\, \partial \bar{z}}$

Vector

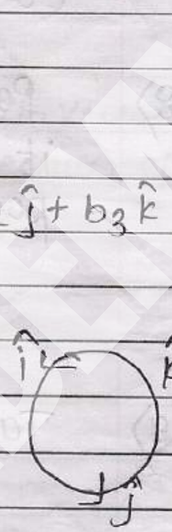
1. Vector (With $\hat{i}, \hat{j}, \hat{k}$)
2. Scalar (Without $\hat{i}, \hat{j}, \hat{k}$)
3. Scalar product (Dot product)

Rule $\begin{cases} \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \end{cases}$

If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

4. Vector Product (Cross product)



$\begin{cases} \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0 \\ \hat{i}\times\hat{j} = \hat{k}, \quad \hat{j}\times\hat{k} = \hat{i}, \quad \hat{k}\times\hat{i} = \hat{j} \\ \hat{j}\times\hat{i} = -k \end{cases}$

$$\bar{a}\times\bar{b} \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

5. Unit Vector $(\wedge) = \dfrac{\text{Vector}}{\text{magnitude}}$

if $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

then $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{a_1\hat{i} + a_2\hat{j} + a_3\hat{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$

6. Scalar Tripple Product

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\cdot\bar{c} = [\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

properties

1) $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}]$

2) $[\bar{a}\ \bar{b}\ \bar{c}] = -[\bar{c}\ \bar{b}\ \bar{a}]$

3) $[\bar{a}\ \bar{b}\ \bar{b}] = 0$

$[\bar{a}\ \bar{b}\ \bar{c}] = 0 \leftarrow$ Vector are coplaner

7) Vector Tripple product
$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\, \bar{b} - (\bar{a} \cdot \bar{b})\, \bar{c}$
$(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a} \cdot \bar{c})\, \bar{b} - (\bar{b} \cdot \bar{c})\, \bar{a}$

8) Position vector ($\bar{r}$)
if $A(x, y, z)$ be any point in plane then it's position vector is .

$$\bar{a} = x\hat{i} + y\hat{j} + z\hat{k} = \bar{r}$$

9) Tangent
$$T = \frac{d\bar{r}}{dt}$$

B) Vector Differentiation

1) Vector Operator (∇)

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

2) Gradient of ϕ
if $\phi(x, y, z)$ be the scalar function then it's gradient is,

$$\text{grad } \phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$
$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

3) Normal to the surface $\bar{N} = \nabla\phi$

4) Angle betn the surface
$$\cos\theta = \hat{N}_1 \cdot \hat{N}_2$$

5) If surface are orthogonal (orthogonal or perpendicular) then $\overline{N}_1 \cdot \overline{N}_2 = 0$

6) Directional Derivative $DD = \nabla\phi \cdot \hat{a}$

↳ Unit vector in the direction of ϕ

7) Max $D.D = |\nabla\phi|$

8) Scalar Potential (ϕ)

$$d\phi = f_1\,dx + f_2\,dy + f_3\,dz$$

9) Work done $= \int \overline{F} \cdot d\overline{r}$

10) Divergence of a vector

if $\overline{F} = f_1\hat{i} + f_2\hat{j} \pm f_3\hat{k}$ be any vector then

$$\text{Div } \overline{F} = \nabla \cdot \overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

if $\nabla \cdot \overline{F} = 0$ then $\overline{F}$ is solenoidal

11) Curl of a vector

If $\overline{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any vector then

$$\text{curl } \overline{F} = \nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

if $\nabla \times \overline{F} = 0$ then $\overline{F}$ is Irrotational (conservative)

c) <u>Vector Integration</u>

1) Green's theorem

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

2) Stoke's theorem

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \bar{N}\cdot(\nabla\times\bar{F})\,\underbrace{ds}_{dx\,dy}$$

3) <u>Gauss Divergence theorem</u>

$$\iint \bar{N}\cdot\bar{F}\,ds = \iiint \nabla\cdot\bar{F}\,dv \rightarrow dx\,dy\,dz$$

$$\nabla\cdot\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$